SCOTCH COLLEGE

MATHEMATICS DEPARTMENT

Year 12 MATHEMATICS SPECIALIST

TEST 2: VECTORS

DATE: 3rd March 2016 Name___________________________

Reading Time: 3 minutes

SECTION ONE: CALCULATOR FREE

TOTAL: 25 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, SCSA formula sheet.

WORKING TIME: 25 minutes (maximum)

SECTION TWO: CALCULATOR ASSUMED

TOTAL: 28 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing instruments, templates, up to 3 Calculators,

1 A4 page of notes (one side only), SCSA formula sheet.

WORKING TIME: 25 minutes (minimum)

SECTION 1 Question	Marks available	Marks awarded	SECTION 2 Question	Marks available	Marks awarded
1	5		6	9	
2	6		7	7	
3	4		8	12	
4	6				
5	4				
Total	25			28	

Section One: Calculator-free **[25 marks]**

This section has **five (5)** questions. Answer **all** questions. Write your answers in the spaces provided

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Question 1 [5 marks]

A straight line passes through the points $P(2,-3)$ and $Q(5,3)$.

(a) Find the vector equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [2]

$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ **(1)**

$\Rightarrow \quad \mathbf{r} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ **OR** Suitable alternative **(1)**

(b) Find the equation of the line through P and Q in parametric form. [1]

$x = 5 + \lambda$

$y = 3 + 2\lambda$ **OR** Suitable alternative **(1)**

(c) Find the equation of the line through P and Q in Cartesian form. [2]

$\lambda = x - 5$

$\lambda = \frac{y-3}{2}$ **(1)**

$\Rightarrow \quad x - 5 = \frac{y-3}{2} \quad \Rightarrow \quad y = 2x - 7$ **(1)**

Question 2 [6 marks]

The point A lies on the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j})$ and the point B has position vector $4\mathbf{i} - 5\mathbf{j}$. Use a method involving a dot product to determine the position vector of A so that the distance from A to B is a minimum. [6]

$$\mathbf{a} = \begin{pmatrix} 2+2\lambda \\ 1-\lambda \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

$$\Rightarrow \quad {}_A\mathbf{r}_B = \begin{pmatrix} 2\lambda-2 \\ 6-\lambda \end{pmatrix}$$ (1)

At point of closest approach

$$\begin{pmatrix} 2\lambda-2 \\ 6-\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0$$ (1)

$$\Rightarrow \quad 4\lambda - 4 - 6 + \lambda = 0$$ (1)

$$\Rightarrow \quad \lambda = 2$$ (1)

$$\Rightarrow \quad \mathbf{a} = \begin{pmatrix} 2+2\times 2 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

(1) (1)

Question 3 [4 marks]

Point A has position vector $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ and point B has position vector $\begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix}$. Find the position vector of the point P that divides AB internally in the ratio $2:3$.

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix}$$ (1)

$$\overrightarrow{AP} = \frac{2}{5}\begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$ (1)

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

(1) (1)

Question 4 [6 marks]

(a) Find a vector perpendicular to the two vectors:

$$\overrightarrow{OP} = \vec{i} - 3\vec{j} + 2\vec{k}$$

$$\overrightarrow{OQ} = -2\vec{i} + \vec{j} - \vec{k}$$ [3]

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \mathbf{i}(1) - \mathbf{j}(-3) + \mathbf{k}(-5) = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

(1) (1) (1)

(b) If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are position vectors for the points P and Q, use your answer to part (a), or otherwise, to find the area of the triangle OPQ. [3]

Area $= \frac{1}{2}|OP| \times |OQ| \times \sin(\theta)$ (1)

$= \frac{1}{2}\left|\overrightarrow{OP} \times \overrightarrow{OQ}\right| = \frac{1}{2}\left|\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}\right| = \frac{\sqrt{35}}{2}$ units^2.

(1) (1)

Question 5 [4 marks]

Points P and Q have coordinates $(3, 1, -2)$ and $(4, 2, -1)$ respectively.

(a) Write a vector equation for the line passing through P and Q. [2]

$$\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1)$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1)$$

(b) Show that the vector $2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ is perpendicular to the line through P and Q. [1]

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0 \quad (1)$$

$$\Rightarrow \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \perp \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

(c) Write down a vector equation of the plane containing P and Q with $2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ as its normal vector. [1]

$$\mathbf{r} \bullet \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{r} \bullet \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 7 \quad (1)$$

NAME:____

Section Two: Calculator-assumed **[25 marks]**

This section has **three (3)** questions. Answer **all** questions. Write your answers in the spaces provided

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Question 6 [9 marks]

Two rockets are fired from different positions at the same time. Rocket 1 leaves from position $-7\boldsymbol{i}+9\boldsymbol{j}-5\boldsymbol{k}$ km at a velocity of $5\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$ km/min and Rocket 2 leaves from position $-6\boldsymbol{i}-5\boldsymbol{j}+2\boldsymbol{k}$ km at a velocity of $9\boldsymbol{i}+6\boldsymbol{j}-3\boldsymbol{k}$ km/min. Each rocket leaves a trail of smoke and, although the rockets do not collide, their smoke trails do intersect.

(a) Find the coordinates of the point at which the smoke trails intersect. [4]

Rocket 1: $\mathbf{r} = \begin{pmatrix} -7 \\ 9 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$

Rocket 2: $\mathbf{r} = \begin{pmatrix} -6 \\ -5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix}$

At point of intersection: $-7+5\lambda=-6+9\mu$ and $9-4\lambda=-5+6\mu$ **(1)**

$\Rightarrow \quad \lambda=2, \quad \mu=1$ **(1)**

This result for λ and μ gives the same z-component of -1. **(1)**

Thus, point of intersection is $(3,1,-1)$ **(1)**

(b) Find the position of Rocket 1 three minutes after firing. [1]

For Rocket 1: $\mathbf{r(3)} = \begin{pmatrix} -7 \\ 9 \\ -5 \end{pmatrix} + 3\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}$ **(1)**

(c) Find the shortest distance of Rocket 1 from the smoke trail of Rocket 2, three minutes after firing. Give your answer to the nearest metre. [4]

For Rocket 1 at $(8,-3,1)$,

$$\overrightarrow{R_2R_1} = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 9\mu-6 \\ 6\mu-5 \\ 3-3\mu \end{pmatrix} = \begin{pmatrix} 14-9\mu \\ 2-6\mu \\ 3\mu-1 \end{pmatrix}$$ **(1)**

Using CAS, $\left|\overrightarrow{R_2R_1}\right|_{MIN}$ = 6.574 km at $t=1.119$ $t=1.119$ minutes.

(1) **(1)**

Thus, shortest distance is 6 574 m. **(1)**

(Can use dot product also, for same result)

Question 7 [7 marks]

(a) The equation of a sphere is given by $x^2+y^2+z^2-6x+4y+8z=153$. Determine the vector equation of the sphere. [3]

$(x-3)^2+(y+2)^2+(z+4)^2=153+9+16+4=182$

(1) (1)

$\Rightarrow$ Equation of sphere is $\left| r-\begin{pmatrix}3\\-2\\-4\end{pmatrix}\right| = \sqrt{182}$. (1)

(b) Determine the position vector(s) of the points of intersection between the sphere and the line $\boldsymbol{r}=-3\boldsymbol{i}+5\boldsymbol{j}+\boldsymbol{k}+\lambda(-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})$. [4]

At point of intersection:

$\left|\begin{pmatrix}-3-2\lambda\\5+\lambda\\1-2\lambda\end{pmatrix}-\begin{pmatrix}3\\-2\\-4\end{pmatrix}\right| = \sqrt{182}$ (1)

$\Rightarrow$ $(-6-2\lambda)^2+(7+\lambda)^2+(5-2\lambda)^2=182$ (1)

$\Rightarrow$ $\lambda=-4,\quad 2$ (1)

$\Rightarrow$ Position vectors of points of intersection are: $\begin{pmatrix}5\\1\\9\end{pmatrix}$ and $\begin{pmatrix}-7\\7\\-3\end{pmatrix}$ (1)

Question 8 [12 marks]

Let $\boldsymbol{r} = \begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix}$, $t \in R$, be an equation of line L.

The plane P has a normal vector $\begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$ and passes through the point $A(-1,0,4)$.

(a) Show that the point $B(9,-5,2)$ lies on the line L. [2]

$2t+5=9 \quad \Rightarrow \quad t=2$ (1)

$$\mathbf{r}(2) = \begin{pmatrix} 2\times 2+5 \\ -2\times 2-1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ 2 \end{pmatrix}$$

$\Rightarrow$ $B(9,-5,2)$ lies on the line L (1)

(b) Give the normal vector equation of the plane P. [2]

$$\mathbf{r} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = -7$$ (1)

$\Rightarrow$ Normal vector equation of P is $\mathbf{r} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = -7$ (1)

(c) Find the shortest distance that plane P is from the origin. [2]

$|\mathbf{n}| = \sqrt{26} \quad \Rightarrow \quad d = \dfrac{7}{\sqrt{26}}$

(1) (1)

(d) Show that the line L meets the plane P at the point $C\,(1, 3, -2)$. [3]

At point of intersection, $\begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = -7$ (1)

$\Rightarrow \quad 6t + 15 + 8t + 4 - t = -7$

$\Rightarrow \quad t = -2$ (1)

$\mathbf{r(-2)} = \begin{pmatrix} 2\times -2+5 \\ -2\times -2-1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ (1)

(e) Find the angle between the line L and the plane P. (Give your answer correct to 1 decimal place.) [3]

Direction of L is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Direction of normal is $\mathbf{n} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$ **(1)**

Angle between L and $\mathbf{n}$ is 31.8^0 **(1)**

Angle between L and P is $90^0 - 31.8^0 = 58.2^0$ **(1)**

END OF QUESTIONS